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Electric and magnetic fields from a semi-infinite antenna above a conducting plane

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Abstract

The electric and magnetic field structures around a semi-infinite thin-wire antenna vertically placed above a perfectly conducting ground plane are investigated when the antenna is supporting two different types of sources. It is shown that when the wire is carrying a uniform line charge, the electrostatic potentials are equal on the surfaces of imaginary cones of fixed cone angles with axis along the wire and apex at the conducting plane. The electrostatic field vectors are shown to be perpendicular to the imaginary cones and tangential to the meridian lines of half-spherical shells centered at the base of the line charge. The vertical components of the electrostatic field on the surface of these imaginary half-spherical shells of a given radius are constant, except at the wire itself. The magnetic field structure associated with a constant current in the semi-infinite antenna is that of an infinite wire. The electric and magnetic fields due to a time-varying charge or current pulse propagating with the speed of light along the vertical thin-wire antenna have a spherical transverse electromagnetic (TEM) field structure, identical to that for the case of a uniform line charge and a uniform current. The connection between the static and dynamic solutions is derived mathematically using two different approaches.

Keywords: Vertical thin-wire antenna; Spherical transverse electromagnetic field; Pulse propagation; Transmission line model; Lightning return stroke

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Introduction

Exact expressions for the electric and magnetic fields from a semi-infinite vertical wire (line) antenna above a ground plane, all conductors being perfect, with a point source at the bottom of the antenna have been recently derived [1,2]. Expressions for the electrostatic fields from an infinitely long and static uniform line charge can be found in the introductory chapters of many electromagnetic text books [3,4]. As far as the authors are aware, electrostatic field expressions for a semi-infinite vertical line charge above a perfectly conducting plane have not been treated in text books. Such a geometry is of practical interest for pulse propagation in wire antennas and for the modelling of lightning return strokes [5,6]. Recently there have been discussions of the equilibrium charge distribution on finite-length conducting wires [7,8], and it has been shown that the distribution tends to be uniform as the wire radius approaches zero. In the present paper, we show that the electric and magnetic field structures around a semi-infinite thin wire antenna carrying a charge or current pulse that propagates up the antenna at the speed of light above a conducting plane are identical to the electrostatic and magnetostatic field structures from a semi-infinite uniform vertical line charge and uniform current above a conducting plane. We derive the dynamic solution, that is the solution for the case of a pulse propagating up the antenna from a point source at the intersection of the antenna and the ground plane, two ways: the first way by assuming that the dynamic fields are spherical transverse electromagnetic (TEM) and by showing that both static and dynamic solutions satisfy the same potential function, and the second way via an exact calculation of the time-varying fields. Throughout this paper the thin-wire antenna is assumed to have a radius that approaches zero. Also, the length of the antenna is semi-infinite for the uniform line charge and current, and semi-infinite or sufficiently long for the pulse propagation so that the pulse front does not reach the end of antenna during the times of interest.

1. Scalar potential and electrostatic field from a semi-infinite uniform line charge vertical to a conducting plane

Consider a uniform vertical line charge having a charge density ρ above a conducting plane. The line charge is along the positive z-axis and the conducting plane is on the x–y plane passing through the origin. The effect of the conducting plane on the electric field can be taken into account by replacing the plane by an image line charge of opposite polarity along the negative z-axis. Let P be an arbitrary point in space. The problem is worked out in spherical coordinates r,θ,ϕ, and the geometry is shown in Fig. 1. The problem has rotational symmetry about the z-axis, and therefore the potential and fields are independent of angle ϕ.

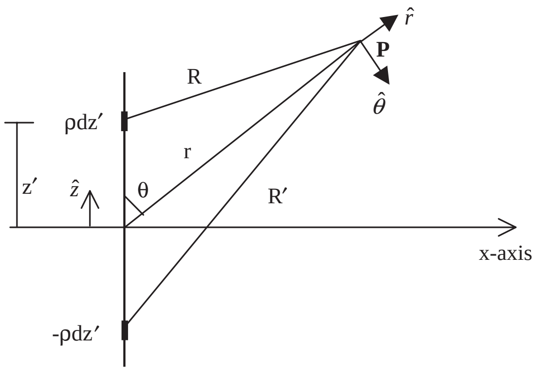

Fig. 1. The geometry for the calculation showing the line charge and its image

1.1. Scalar potential

The scalar potential at P due to an elemental charge $\rho\,\mathrm{d}z'$ is

$$V_1(r,\theta)=\int_0^L \frac{\rho\,\mathrm{d}z'}{4\pi\varepsilon_0 R}, \tag{1}$$

where $R=\sqrt{r^2+z'^2-2z'r\cos\theta}$, and L is the length of the line charge.

Integrating (1), we find

$$\begin{aligned}V_1(r,\theta) &= \frac{\rho}{4\pi\varepsilon_0}\left|\mathrm{Sinh}^{-1}\frac{2z'-2r\cos\theta}{\sqrt{4r^2(1-\cos^2\theta)}}\right|_0^L\\ &= \frac{\rho}{4\pi\varepsilon_0}\left[\mathrm{Sinh}^{-1}\frac{L-r\cos\theta}{r\sin\theta}+\mathrm{Sinh}^{-1}\frac{\cos\theta}{\sin\theta}\right].\end{aligned} \tag{2}$$

The influence of perfectly conducting ground plane can be taken into account by considering an image line charge of equal magnitude, but opposite polarity. The scalar potential from the image line charge is given by

$$V_2(r,\theta)=\int_0^L \frac{-\rho\,\mathrm{d}z'}{4\pi\varepsilon_0 R'}, \tag{3}$$

where $R'=\sqrt{r^2+z'^2-2z'r\cos(\pi-\theta)}$.

Integrating (3), we find

$$\begin{aligned}V_2(r,\theta) &= \frac{-\rho}{4\pi\varepsilon_0}\left|\mathrm{Sinh}^{-1}\frac{2z'+2r\cos\theta}{\sqrt{4r^2(1-\cos^2\theta)}}\right|_0^L\\ &= \frac{-\rho}{4\pi\varepsilon_0}\left[\mathrm{Sinh}^{-1}\frac{L+r\cos\theta}{r\sin\theta}-\mathrm{Sinh}^{-1}\frac{\cos\theta}{\sin\theta}\right].\end{aligned} \tag{4}$$

Adding (2) and (4), we obtain the total scalar potential as

$$V(r,\theta) = \frac{\rho}{4\pi\varepsilon_0}\left[\mathrm{Sinh}^{-1}\frac{L - r\cos\theta}{r\sin\theta} - \mathrm{Sinh}^{-1}\frac{L + r\cos\theta}{r\sin\theta}\right] + \frac{\rho}{4\pi\varepsilon_0}\left[\mathrm{Sinh}^{-1}\frac{\cos\theta}{\sin\theta} + \mathrm{Sinh}^{-1}\frac{\cos\theta}{\sin\theta}\right]. \quad (5)$$

Applying the relation $\mathrm{Sinh}^{-1}\, x = \ln(x + \sqrt{x^2+1}), -\infty < x < \infty$ to (5) and simplifying, we find

$$V(r,\theta) = \frac{\rho}{4\pi\varepsilon_0}\left[\ln\frac{L - r\cos\theta + \sqrt{L^2 + r^2 - 2L\,r\cos\theta}}{L + r\cos\theta + \sqrt{L^2 + r^2 + 2L\,r\cos\theta}}\right] + \frac{\rho}{2\pi\varepsilon_0}\left[\ln\frac{1+\cos\theta}{\sin\theta}\right]. \quad (6)$$

In the limit that L approaches infinity, it can be shown that the first term of (6) becomes zero ($\ln 1 = 0$). Therefore for an infinite line charge (6) reduces to

$$V(\theta) = \frac{\rho}{2\pi\varepsilon_0}\left[\ln\frac{1+\cos\theta}{\sin\theta}\right]. \quad (7)$$

Alternatively, (7) can also be written as

$$V(\theta) = \frac{\rho}{4\pi\varepsilon_0}\left[\ln\frac{1+\cos\theta}{1-\cos\theta}\right]. \quad (7a)$$

From (7) it is clear that the scalar potential is dependent only on the angle θ, and is independent of the distance r from the origin. The equipotential surfaces are surfaces of cones whose apex is at the origin and whose axis is along the line charge. This is illustrated in Fig. 2. The magnitude of the potential is zero at the conducting plane ($\theta = 90^\circ$) and increases with decreasing angle, and becomes infinite on the line charge ($\theta = 0$). Eq. (7), away from the source charge, satisfies Laplace's equation

$$\nabla^2 V = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0,$$

since V is only a function of θ and since

$$\frac{\partial V}{\partial\theta} = -\frac{\rho}{2\pi\varepsilon_0\sin\theta}.$$

1.2. Electrostatic field

The electrostatic field at P due to the semi-infinite line charge above the conducting plane is given by

$$\bar{E} = -\nabla V = -\hat{\theta}\frac{1}{r}\frac{\partial V}{\partial\theta}, \quad (8)$$

since V is not a function of r or ϕ.

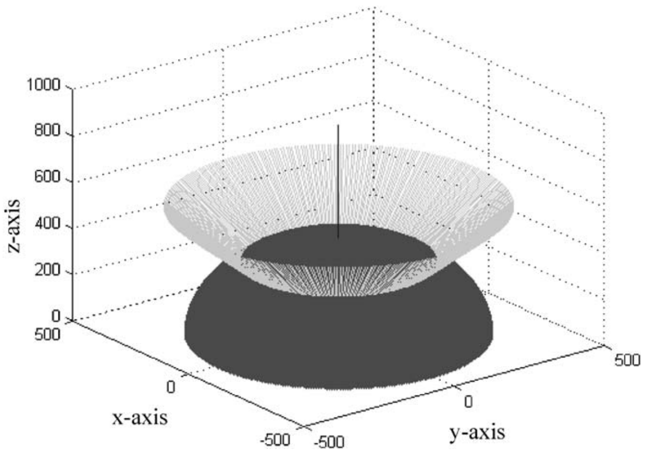

Fig. 2. Illustration of electrostatic potential and electrostatic field structure from a semi-infinite line charge above a perfectly conducting ground plane. The potential is equal at the surface of the cone (constant apex angle θ) and the electric field vectors are tangential to the hemi-spherical surface (constant radius r). The cone and hemispherical surface intersect at right angles and the electric field vector is constant on the circle formed at the junction of these two surfaces (constant θ and r). The electric field vector normal to the ground plane has constant magnitude on the hemi-spherical surface. The scales are arbitrary in units of distance.

Evaluating (8) and simplifying, we find

$$\bar{E}(r,\theta) = \hat{\theta}\frac{\rho}{2\pi\varepsilon_0 r \sin\theta}. \tag{9}$$

From (9) it is apparent that the electrostatic field has only a θ-component and is tangential to a semi-spherical shell of radius r (see Fig. 2). On the conducting plane the electric field is perpendicular, as expected, and is a minimum. The electric field vector in the θ-direction on the semi-spherical shell increases with decreasing θ and becomes infinite on the line charge. The cone and hemispherical surface intersect at right angles and the electric field vector is constant on the circle formed at the junction of these two surfaces. Also, it can be easily shown that the vertical component of the electric field anywhere on the semi-spherical shell is a constant and is given by

$$\bar{E}_z(r) = -\hat{z}\frac{\rho}{2\pi\varepsilon_0 r} \tag{10}$$

except at the wire itself corresponding to $\theta = 0$.

1.3. Magnetostatic field

Let us now replace the line charge by a line current I, directed vertically upward (z-direction). The effect of the conducting plane on the magnetic field can be obtained by replacing the ground plane by an image line current. The direction of the image line current is the same as that of the original line current. As the length of the line current is made infinite, the problem is reduced to an infinite line current I. From Ampere's law, the magnetic field at P in cylindrical or rectangular co-ordinates is given by several text books (e.g. [3]), and in spherical co-ordinates it can be written as

$$\bar{B}(r,\theta)=\hat{\phi}\frac{\mu_0 I}{2\pi r\sin\theta}=\hat{\phi}\frac{I}{2\pi\varepsilon_0 c^2 r\sin\theta}, \tag{11}$$

where $r\sin\theta$ is the distance from the line current to the point P.

2. Pulse propagation on a vertical wire above a conducting plane: intuitive approach

Consider a vertical conductor above a conducting ground plane, all conductors being perfect, that is, having infinite conductivity. As in the static line charge considered above, the conductor is assumed to have a radius that approaches zero. Assume a point source that varies with time at the bottom between the vertical conductor and the plane. The source will cause a time varying current and charge distribution on the vertical conductor and an associated electric and magnetic fields will be set up in space. The vertical conductor is assumed to be sufficiently long so that the wavefront of the current pulse or electromagnetic waves has not reached the end of the conductor within time t of interest. The overall problem is related to pulse propagation in long monopole antennas and in lightning return strokes. We want to find the solution for the electric and magnetic fields in terms of the source. The problem has axial symmetry and therefore the fields cannot vary with the azimuth angle ϕ. At time $t=0$, the only source in the problem space is a point charge at the bottom of the vertical conductor. The boundary conditions on perfect conductors require that the electric fields at the ground plane and at the vertical conductor be normal and that the magnetic field has only a component in the ϕ-direction. Electric field vectors tangential to hemispherical shells of radius r and centered at the point source at the origin satisfy those boundary conditions suggesting that the field structure could be in the form of a spherical TEM wave.

Let us now try to find a general expression for the electric and magnetic fields, starting from the assumption of a spherical TEM wave. If the resulting fields satisfy the wave equation and all the boundary conditions (i.e. satisfy Maxwell's equations subject to boundary conditions), then from the uniqueness theorem they are indeed the only solutions. For a spherical TEM wave with ϕ-symmetry, electric and magnetic fields are of the form $E(r,\theta,t-r/c)$ and $B(r,\theta,t-r/c)$, respectively. The curl of E and B fields can be separated into two parts, one part corresponding to the transverse surface and other corresponding to the direction of propagation normal to the surface. That is, for example, $\nabla\times\bar{E}=\nabla_T\times\bar{E}+\nabla_r\times\bar{E}$. Further, $E_r=B_r=0$ since all fields are assumed perpendicular to the direction of propagation r (TEM assumption), so that $\nabla_T\times\bar{E}=0$ and $\nabla_T\times\bar{B}=0$. The fact that $\nabla_T\times\bar{E}=0$ and that $\nabla_T\cdot\bar{E}=0$ away from the sources allows the electric field on a transverse (spherical) surface to be derived from the gradient of a potential that satisfies a two-dimensional Laplace's equation on the transverse surface. Since the electric field vector has only a θ-component, and the electric field on the transverse surface is determined by the gradient of a scalar potential, it is clear that the scalar potential V can be only a function of θ. Thus, the appropriate Laplacian is given by

$$\nabla_T^2 V(\theta) = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V(\theta)}{\partial\theta}\right) = 0, \tag{12}$$

exactly the same equation satisfied by the three-dimensional electrostatic field of the semi-infinite line charge discussed in the previous section. Simplifying (12), we obtain the equation

$$\frac{\partial^2 V(\theta)}{\partial\theta} + \cot\theta \frac{\partial V(\theta)}{\partial\theta} = 0 \tag{13}$$

which has a general solution of the form

$$V(\theta) = A + C \ln\left(\cot\frac{\theta}{2}\right), \tag{14}$$

where A and C are constants to be determined from the boundary conditions. C is clearly a function of the source. The scalar potential is zero at the conducting plane, i.e. when $\theta = 90°$. Applying this condition to (14), we see that $A = 0$. Therefore,

$$V(\theta) = C \ln(\cot\frac{\theta}{2}) = -C \ln(\tan\frac{\theta}{2}) = \frac{C}{2} \ln\frac{1+\cos\theta}{1-\cos\theta}. \tag{15}$$

Eq. (15), as expected from the argument above, is of the same form as (7a), the expression for the potential from a uniform line charge above conducting plane. As noted above, for the TEM wave, the electric field is determined by the gradient of the scalar potential. Therefore from (15), we find

$$\bar{E} = -\nabla V(\theta) = -\hat{\theta}\frac{1}{r}\frac{\partial V(\theta)}{\partial\theta} = +\hat{\theta}\frac{C}{r}\frac{\partial}{\partial\theta}\ln(\tan\frac{\theta}{2}) = \hat{\theta}\frac{C}{r\sin\theta}. \tag{16}$$

Eq. (16), as expected, is of the same form as (9), the expression for the electrostatic field from a vertical uniform line charge placed above a conducting plane. The multiplier C in (16) is evaluated later in this section.

For a TEM wave, $\nabla_T \times \bar{B} = 0$. That is,

$$\frac{\hat{r}}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta \cdot B_\phi)\right) = 0$$

which implies that

$$\frac{\partial}{\partial\theta}(\sin\theta \cdot B_\phi) = 0. \tag{17}$$

Eq. (17) has a solution of the form,

$$B_\phi = \frac{F}{\sin\theta}, \tag{18}$$

where the multiplier F is independent of θ and has to be determined. A TEM solution implies that the current released from the point source at the bottom of the thin wire travels up the antenna at the speed of light along with the spherical wavefront of the electromagnetic fields. That is, there is no time retardation between the current at a point on the antenna and the fields anywhere on the hemispherical surface passing through this point. For a TEM wave, Ampere's law is valid on the surface of a sphere of radius r. F is evaluated by applying Ampere's law, that is, integrating B_ϕ along a path around the vertical antenna on the surface of the TEM hemi-sphere as θ approached zero. At that point, the current enclosed by the line integral is the current at the intersection point of the antenna and the sphere. Therefore, the complete magnetic field in the time varying case is given by

$$\bar{B} = \hat{\phi} B_\phi = \hat{\phi} \frac{i(t - r/c)}{2\pi\varepsilon_0 c^2 r \sin\theta}, \tag{19}$$

where $i(t-r/c)$ is the retarded current at the source. From B, E can be obtained as follows:

$$\nabla \times \bar{B} = \nabla_T \times \bar{B} + \frac{\hat{\theta}}{r}\left(-\frac{\partial}{\partial r}(r \cdot B_\phi)\right) = \frac{1}{c^2}\frac{\partial \bar{E}}{\partial t}. \tag{20}$$

Substituting (19) in (20) and noting that $\nabla_T \times \bar{B} = 0$, we can rewrite (20) as

$$\frac{\hat{\theta}}{r}\left(-\frac{1}{2\pi\varepsilon_0 c^2 \sin\theta} \cdot (-\frac{1}{c}) \cdot \frac{\partial i(t - r/c)}{\partial t}\right) = \frac{1}{c^2}\frac{\partial \bar{E}}{\partial t}. \tag{21}$$

Simplifying and integrating (21), and noting that $i(0,0) = 0$, we find

$$\bar{E} = \hat{\theta} \frac{i(t - r/c)}{2\pi\varepsilon_0 c r \sin\theta}. \tag{22}$$

Comparing (22) and (16), we obtain $C = i(t - r/c)/(2\pi\varepsilon_0 c)$.

Eqs. (19) and (22) represent the complete solution for E and B in the time varying case. From the continuity equation, we find

$$\rho(t - r/c) = \frac{i(t - r/c)}{c}. \tag{23}$$

Eq. (23) shows that the current is proportional to charge, and further confirms that the current released from the point source travels unattenuated with the speed of light along the vertical conductor, effectively creating a line charge density which goes to zero at those points where the current is zero.

Substituting (23) in (22), we find E can alternatively be written as

$$\bar{E} = \hat{\theta} \frac{\rho(t - r/c)}{2\pi\varepsilon_0 r \sin\theta}. \tag{24}$$

The ratio between E and B, from (22) and (19) is c, the speed of light, consistent with a TEM wave. The wave impedance is the free-space impedance at all distances from the antenna.

It can be easily verified that (19) and (22) satisfy the three-dimensional wave equations

$$\nabla^2 \bar{E} - \frac{1}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2} = 0 \tag{25}$$

and

$$\nabla^2 \bar{B} - \frac{1}{c^2}\frac{\partial^2 \bar{B}}{\partial t^2} = 0, \tag{26}$$

respectively, which further confirms that E and B given by (22) and (19) are indeed the complete and correct expressions.

The Poynting vector, the cross product of (22) and (19), is in the r-direction, which indicates energy flow in the radial direction from the source at the bottom of the antenna. That is, in this ideal case the only source of radiation is the point source at the bottom of the antenna and the vertical antenna itself does not radiate. However, the situation is different if the pulse reaches and reflects from the top of a finite antenna. That case is not considered here.

3. Pulse propagation on a vertical wire antenna above a conducting plane: exact formulation

The exact expression for the electric and magnetic fields from a time-varying current or charge density distribution behind an upward traveling lightning return stroke wavefront is derived in [5,9–12]. The problem is similar to a linear current or charge distribution fixed at one end and extending continuously at the other end with some speed. The geometry of the problem and the electric and magnetic field expressions from [11] are reproduced in the Appendix. It is shown that if the source is a current pulse that travels upward with the speed of light and without any attenuation and dispersion, then the exact expressions for electric field and magnetic field, given in (A.4) and (A.5), respectively, from that part of the antenna above the ground plane (not including the effects of the ground plane) reduces to simple expressions [1] given by

$$\bar{E}(r,t) = \frac{-1}{4\pi\varepsilon_0 r^2}\int_{r/c}^{t} i(0,\tau - r/c)\,\mathrm{d}\tau\hat{r} + \frac{(1+\cos\theta)}{4\pi\varepsilon_0 cr\sin\theta} i(0,t-r/c)\hat{\theta}, \quad \theta \neq 0, \tag{27}$$

$$B_\phi(r,\theta,t) = \frac{1+\cos\theta}{4\pi\varepsilon_0 c^2 r\sin\theta} i(0,t-r/c), \quad \theta \neq 0. \tag{28}$$

The first term in (27) is independent of the angle θ and can be viewed as the field created due to the depletion of a point charge at the base of the channel. If the vertical line is above a perfectly conducting ground plane, then the effect of the ground plane on the fields above can be taken into account by replacing the ground plane by an image line perpendicular to the ground plane and beneath it carrying an equal current in the same direction. Adding to the fields in (27) and (28), the contributions from the image channel, we obtain the expressions for total electric and magnetic fields [1] as

$$\bar{E}(r,\theta,t) = \frac{1}{2\pi\varepsilon_0 cr\sin\theta} i(0,t-r/c)\hat{\theta}, \theta \neq 0, \tag{29}$$

$$\bar{B}(r,\theta,t) = \frac{1}{2\pi\varepsilon_0 c^2 r \sin\theta} i(0, t - r/c)\hat{\phi}, \theta \neq 0. \quad (30)$$

These expressions are the same as (22) and (19), respectively, found in the previous section using a TEM wave assumption, and can be considered as a justification of that approach.

The unique and exact solution above (Eqs. (29) and (30)) for a current pulse traveling up a vertical antenna above a conducting plane, all conductors being perfect, can also be obtained from the solution of the wave equation between two concentric conical surfaces of infinite conductivity with common cone apexes [13], the case of a vertical antenna above a ground plane being viewed as the limiting case in which the polar angle of one of the cones goes to zero (the vertical wire) and the polar angle of the other cone goes to 90° (the ground plane). If the only charge/current injection point in the problem space is a point source located at the base of the vertical conductor of vanishing radius, then the electromagnetic field created between the conductor and ground can only have a spherical TEM structure originating at the charge source and expanding outwards in the r-direction with the speed of light.

Conclusions

In this paper, we have shown that electric and magnetic fields from a semi-infinite vertical thin-wire antenna above a conducting plane, carrying uniform charge or current, are tangential to the hemi-spherical shells of radius r, centered at origin at the bottom of the wire. It is shown that an identical field structure, a spherical TEM structure, is produced by a propagating pulse charge and current due to a point source at the bottom of the thin-wire (radius approaching zero) antenna. The relationships between the static and dynamic solutions are derived mathematically.

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Appendix. A

Exact expression for electric and magnetic fields from a charge or current distribution extending upward from point A.

The following general expressions for the electric and magnetic fields were originally derived for the lightning return-stroke channel [10–12]. The return-stroke channel can be modelled as a straight line fixed at one end A, with the other end extending with some speed. The geometry of the problem is shown in Fig. 3. The current on the lightning channel is represented by $i(z', t)$, where z' indicates the position along the z-axis with origin at the base of the channel and t indicates the time. At time $t = 0$ the return stroke starts to propagate from origin A. The observer at the fixed field point P 'sees' the return stroke starting to propagate from the origin at time $t=r/c$, where c is the speed of light. The retarded current at any elemental channel section dz' is given by $i(z', t - R(z')/c)$, where z' is less than or equal to $L'(t)$, the length of the return stroke channel 'seen' by the observer at P at time t. The vector potential at P due to the entire extending channel is given by

$$\bar{A}(r, \theta, \tau) = \frac{1}{4\pi\varepsilon_0 c^2} \int_0^{L'(\tau)} \frac{i(z', \tau - R(z'/c)}{R(z')} \hat{z}\, dz', \tag{A.1}$$

where τ is a time less than or equal to time t. At time τ, return-stroke wavefront is "seen" at a height $L'(\tau)$ by the observer at P and $L'(\tau)$ is less than or equal to $L'(t)$. Note that in Eq. (A.1) we have not considered the presence of ground, usually assumed to be perfectly conducting and replaced by a channel image.

The total electric field can be calculated using the relation

$$\bar{E} = -\nabla\phi - \frac{\partial \bar{A}}{\partial t}, \tag{A.2}$$

where ϕ can be obtained from the Lorentz condition

$$\nabla \bullet \bar{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0,$$

as

$$\phi(r, \theta, t) = -c^2 \int_{r/c}^{t} \nabla \bullet \bar{A}\, \mathrm{d}\tau. \tag{A.3}$$

Substituting (A.1) and (A.3) in (A.2), the electric field is given by [11]

$$\bar{E}(r, \theta, t) = -\frac{1}{4\pi\varepsilon_0}\hat{r} \int_0^{L'(t)} \frac{\cos\theta - 3\cos\alpha(z')\cos\beta(z')}{R^3(z')} \int_{t_b}^{t} i\left(z', \tau - \frac{R(z')}{c}\right) \mathrm{d}\tau\, \mathrm{d}z', \tag{A.4a}$$

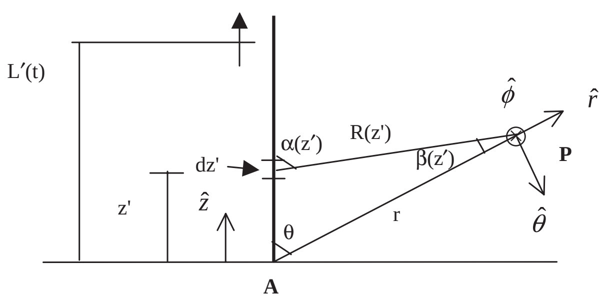

Fig. 3. Geometry of the problem used to derive field expressions (A.4) and (A.5) (adapted from [11]).

$$-\frac{1}{4\pi\varepsilon_0}\hat{r}\int_0^{L'(t)}\frac{\cos\theta-3\cos\alpha(z')\cos\beta(z')}{cR^2(z')}i\left(z',t-\frac{R(z')}{c}\right)\,\mathrm{d}z', \tag{A.4b}$$

$$-\frac{1}{4\pi\varepsilon_0}\hat{r}\int_0^{L'(t)}\frac{\cos\theta-\cos\alpha(z')\cos\beta(z')}{c^2R(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}\,\mathrm{d}z', \tag{A.4c}$$

$$+\frac{1}{4\pi\varepsilon_0}\hat{\theta}\int_0^{L'(t)}\frac{\sin\theta+3\cos\alpha(z')\sin\beta(z')}{R^3(z')}\int_{t_b}^{t}i(z',\tau-\frac{R(z')}{c})\,\mathrm{d}\tau\mathrm{d}z', \tag{A.4d}$$

$$+\frac{1}{4\pi\varepsilon_0}\hat{\theta}\int_0^{L'(t)}\frac{\sin\theta+3\cos\alpha(z')\sin\beta(z')}{cR^2(z')}i\left(z',t-\frac{R(z')}{c}\right)\,\mathrm{d}z', \tag{A.4e}$$

$$+\frac{1}{4\pi\varepsilon_0}\hat{\theta}\int_0^{L'(t)}\frac{\sin\theta+\cos\alpha(z')\sin\beta(z')}{c^2R(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}\,\mathrm{d}z', \tag{A.4f}$$

$$-\frac{1}{4\pi\varepsilon_0}\hat{r}\frac{\cos\theta-\cos\alpha(L')\cos\beta(L')}{c^2R(L')}i\left(L',t-\frac{R(L')}{c}\right)\frac{\mathrm{d}L'}{\mathrm{d}t}, \tag{A.4g}$$

$$+\frac{1}{4\pi\varepsilon_0}\hat{\theta}\frac{\sin\theta+\cos\alpha(L')\sin\beta(L')}{c^2R(L')}i\left(L',t-\frac{R(L')}{c}\right)\frac{\mathrm{d}L'}{\mathrm{d}t}. \tag{A.4h}$$

In Eq. (A.4), $\mathrm{d}L'/\mathrm{d}t$ is the speed of the current wavefront as 'seen' by the observer at P, which is different from the real speed. Also, from Fig. 3 we get $\cos\alpha(z')=-(z'-r\cos\theta)/R(z')$, $\cos\beta(z')=(r-z'\cos\theta)/R(z')$, and $\sin\beta(z')=z'\sin\theta/R(z')$. The lower limit of the time integral of the first term in (A.4), t_b, is the time at which the return-stroke wavefront has reached the height z' for the first time, as 'seen' from the observation point. The last two terms of expression (A.4) containing $\mathrm{d}L'/\mathrm{d}t$ will have non-zero values only if there is a current discontinuity (non-zero current) at the wavefront.

A.1. Magnetic field

The magnetic field is given by $\bar{B}=\nabla\times\bar{A}$. For a vertical channel the magnetic field has only a horizontal component and it is given by [5]

$$\begin{aligned}B(r,t)&=\frac{1}{4\pi\varepsilon_0c^2}\hat{\phi}\int_0^{L'(t)}\left(\frac{\sin\alpha(z')}{R^2(z')}i(z',t-R(z')/c+\frac{\sin\alpha(z')\partial i(z',t-R(z')/c)}{cR(z')\qquad\partial t}\right)\mathrm{d}z'\\&\quad+\frac{1}{4\pi\varepsilon_0c^2}\hat{\phi}\frac{\sin\alpha(L')}{cR(L')}i(L',t-R(L')/c)\frac{\mathrm{d}L'}{\mathrm{d}t},\end{aligned} \tag{A.5}$$

where $\sin\alpha(z')=r\sin\theta/R(z')$

Eqs. (A.4) and (A.5) are valid for any return-stroke model. In the electric field expression (A.4) terms containing the factors R^{-3},$c^{-1}R^{-2}$, and $c^{-2}R^{-1}$ are called the static component, the induction component, and the radiation component, respectively.

In the transmission line model of the return stroke, the return stroke current pulse is considered to propagate upward without attenuation and dispersion. In such a case, and for propagation at the speed of light, the retarded current at any position z' along the vertical channel is related to the current at the base of the channel by the expression

$$i\left(z', t - \frac{R(z')}{c}\right) = i\left(0, t - \frac{z'}{c} - \frac{R(z')}{c}\right). \tag{A.6}$$

Substituting (A.6) in (A.4) and (A.5) and doing several mathematical manipulations, it can be shown that (A.4) and (A.5) identically reduces to the simple expressions (27) and (28), respectively.

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